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On quantum spin-chain spectra and the representation theory of Hecke algebras by augmented braid diagrams

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Abstract. We use simple diagrammatic techniques to analyse the ordinary representation theory of the A_n Hecke algebras $H_n(q)$, and to construct $H_n(q)$ modules (resp. representations) which are generically simple (irreducible) and well defined in every specialization of q, including roots of unity. We determine several physically important properties of these modules, generalizing properties of the Temperley–Lieb algebra and *its* diagrams which have proved useful for lattice models.

We show how these results can be used to locate energy level crossings in $U_q\,\mathrm{sl}(N)$ invariant quantum spin chains, and locate a new crossing of the thermodynamic limit $U_q\,\mathrm{sl}(3)$ spin chain at $\frac{q^5-1}{q-1} = 0$ as an example.

1. Introduction

Recent results of Lascoux *et al* [29], Grojnowski [20], Soergel [43] and Ariki [2] narrow a crucial gap in the programme of analysis of A_n Hecke algebra representation theory proposed in [32]. The programme can now be developed to give extensive information on the energy level crossings of $U_q\,\mathrm{sl}(N)$ invariant quantum spin chains and vertex models (cf [41, 42]). This note begins with a review of the relevant background in the ordinary representation theory of the A_n Hecke algebras $H_n(q)$, using a simple diagrammatic technique (a variant of braids [5] with some features of Penrose diagrams [39]). In fact this technique turns out to be so powerful that we are able to include quick and simple new proofs of key, but hitherto difficult, results in our review. We then proceed to give new results on the structure of the exceptional cases. We use these to show how 'classical' results for the symmetric group may be applied to spectrum level crossing problems, concluding with some specific results in this area. Taken with the recent work of Lascoux *et al* [29] and Ariki [2] on decomposition matrices of *Specht modules* this should in principle enable the reader to compute all q-variation level crossings of $U_q\,\mathrm{sl}(N)$ spin chains.

The Hecke algebras are important in the study of exactly solvable models [4, 9, 10], in computation in more general statistical mechanical models, and in the study of reaction-diffusion processes [1]. They arise particularly in quantum spin chains [40] (including those thought to be relevant to Anderson's t–J model, and hence to high T_c superconductivity [3, 27, 34], although we will not make our approach specific to this case) and in the transfer-matrix formulation of many classical two-dimensional models such as Potts, vertex, IRF and generalized Andrews–Baxter–Forrester models. For a review and references see [12]. The 'universal' Hamiltonian, $\mathcal{H}$, is given in equation (6) below. In all cases $H_n(q)$ for

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given n builds a system of physical 'size' proportional to n, so it is rather the sequence of algebras

$$H_* = \{H_1(q) \subset \cdots \subset H_n(q) \subset H_{n+1}(q) \subset \cdots\}$$

approaching the thermodynamic limit of large n which is of interest. The collective representation theory of these algebras gives a classification scheme for the spectra of spin chains [34] and for classical thermodynamic observables [32], and may reveal hidden symmetries of the model [38]. It also determines a lower bound on the degeneracies of eigenvalues in the Hamiltonian $\mathcal{H}$, or transfer matrix $\mathcal{T}$, and $q =$ roots of unity have been identified as energy level crossing points ([33] and see later).

Unfortunately the determination of the structure of $H_n(q)$ at roots of unity has always appeared very complicated. (Even a *description* of the structure, where known, can appear so [32]!) Here, by pulling together several strands of recent technology and illustrating them with some intuitive pictures we have been able to develop a relatively straightforward form of analysis (and one which provides a suitably unified treatment of all the elements of H_*). This simplicity has in turn allowed us to obtain previously inaccessible results on the structure of the $q^r = 1$ exceptional cases.

1.1. Basic definitions

(1.1). Recall [5, 22] that the *braid group* on n strings B_n has generators $1, g_1, g_2, \ldots, g_{n-1}$ and inverses, and relations:

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \tag{1}$$

$$g_i g_j = g_j g_i \qquad i \neq j \pm 1. \tag{2}$$

(1.2). For $q \in \mathbb{C}$ the Hecke algebra $H_n = H_n(q)$ is the quotient of the braid group algebra $\mathbb{C}B_n$ by the relations

$$(g_i + 1)(g_i - q) = 0. \tag{3}$$

Note in particular, therefore, that $H_n(1) = \mathbb{C}S_n$, the group algebra of the symmetric group. Recall that this is a semi-simple algebra with irreducible representations indexed by *Young diagrams* of degree n, written $\lambda \vdash n$ [24].

(1.3). The set of *weights* of degree n, depth N, is

$$\Lambda(n, N) = \left\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) | \lambda_i \in \mathbb{N}_0, \sum_i \lambda_i = n\right\}.$$

Let S_N act on $\Lambda(n, N)$ via $\pi\lambda = (\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots)$. The orbits of this action may be indexed by their *dominant weights* (Young diagrams)

$$\Lambda^+(n, N) = \{\lambda \in \Lambda(n, N) | \lambda_i \geqslant \lambda_{i+1}\}$$

so $\lambda \vdash n$ means $\lambda \in \Lambda^+(n, n)$. For $\lambda \in \Lambda(n, N)$ let λ^d denote the dominant weight in the orbit of λ. For $\lambda \vdash n$ let λ' denote the Young diagram *conjugate* to λ.

(1.4). A set of words in $1, g_1, g_2, \ldots, g_{n-1}$ is a basis of $H_n(q)$ if it is identically the set S_n in case $q = 1$. Conversely, writing S_n as a set of reduced words, this set passes to a *reduced* basis $\mathfrak{B}_n$ of $H_n(q)$, unique up to equations (1) and (2).

(1.5). By equation (3) *every* word is in $\mathbb{H}_n := \mathbb{Z}[q]\mathfrak{B}_n$, the $\mathbb{Z}[q]$ span of the reduced basis. Thus $\mathbb{H}_n$ is a $\mathbb{Z}[q]$-form of the algebra, and $H_n(q) := \mathbb{C} \otimes_{\mathbb{Z}[q]} \mathbb{H}_n$.

(1.6). For $x \in H_n(q)$ and $b \in \mathfrak{B}_n$ write $\mathfrak{C}_b(x)$ for the coefficient of b in x.

(1.7). For $b \in \mathfrak{B}_n$ let b^T denote the word obtained by reversing the order of letters. This transformation may be extended linearly to $H_n(q)$.

(1.8). For $x \in H_n(q)$ then $x^{(k)}$ is x 'translated' by $g_i \mapsto g_{i+k}$ for all i (we think of $H_n(q)$ embedded naturally in $H_m(q)$, $m \gg n, k$).

The main outstanding interest in Hecke algebras lies in cases where $H_n(q)$ has a representation theory different from that of $\mathbb{C}S_n$. As we will verify, $H_n(q) \cong \mathbb{C}S_n$ unless q is one of a certain set of algebraic points. We will call these points *special*, and the remainder *generic*.

(1.9). Irrespective of the choice of q, we always have a q-generalization of the (unnormalized) Young symmetrizer [47]. For $N \in \mathbb{N}$

$$L_{N+1} = 1 + g_1 + g_2 g_1 + \cdots + g_N g_{N-1} \ldots g_1 \tag{4}$$

(so $L_3^{(1)} = 1 + g_2 + g_3 g_2$, for example); and the unnormalized q-Young symmetrizer is given by

$$Y_0 = Y_1 = 1 \qquad Y_{N+1} = L_{N+1} Y_N^{(1)}. \tag{5}$$

(1.10). Let $[N] = \frac{q^N - 1}{q-1}$ (called q-integers), $[0]! = 1$, $[N]! = \prod_{i=1}^{N} [i]$, and for $\lambda \vdash n$ $[\lambda]! = \prod_i [\lambda_i]!$ (product over rows of λ). Let K denote the monoid generated by q-integers and q.

Proposition 1. The element $Y_N \in \mathbb{H}_N$ obeys $Y_N = \sum_{b \in \mathfrak{B}_n} b$, $Y_N^T = Y_N$ and

$$g_i Y_N = Y_N g_i = q Y_N \qquad (i = 1, 2, \ldots, N-1)$$
$$Y_N Y_N = [N]! Y_N.$$

(1.11). Hence for $1 \leqslant m < n$, $H_n Y_m H_n \supset H_n Y_{m+1} H_n$. For $N \in \mathbb{N}$ define quotient algebras

$$H_n^N := \begin{cases} H_n / H_n Y_{N+1} H_n & N < n \\ H_n & N \geqslant n \end{cases}.$$

(1.12). In a given physical model the representation $\mathcal{R}$ of $H_n(q)$ appearing in the model Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{n-1} \mathcal{R}(g_i) \tag{6}$$

(or the corresponding transfer matrix) is universally characterized by the vanishing of the image of certain elements of the algebra. For example, for V_N the fundamental $U_q\,\mathrm{sl}(N)$ module, the representation $\mathcal{R}_N : H_n(q) \to \mathrm{End}(V_N^{\otimes n})$ which appears in the $U_q\,\mathrm{sl}(N)$ invariant spin chain and associated vertex models [10, 11, 33] obeys $\mathcal{R}_N(Y_{N+1}) = 0$. Indeed the quotient algebras H_n^N are *faithfully* represented by the representations $\mathcal{R}_N$ arising in these models [33]. Now while for given n the algebras H_n and H_n^N can be identified for large enough N, with fixed N the sequence H_* and the sequence

$$H_*^N = \{H_n^N : n = 1, 2, \ldots; N \text{ fixed}\}$$

are markedly different objects (for example, the corresponding Hamiltonians have different ground states [34]).

1.2. Representation theory generalities

We will show that the representation theory of H_n^N may be determined largely from that of the subalgebras H_m^N for all $m < n$, with the remaining calculations being relatively simple. This means in principle that any H_n^N can be analysed by iteration on n from the base $H_1 = H_1^N \cong \mathbb{C}$.

In particular, one crucial observation relates H_*^N algebras of different size n:

Proposition 2. For $q, [N]! \neq 0$ there is an isomorphism of unital algebras

$$Y_N H_{n+N}^N Y_N \cong H_n^N$$

given by $x^{(N)} Y_N \mapsto x$.

We will prove this diagrammatically in section 2. Let us first look at the consequences of this result. To do this succinctly we can use category theory [7] (readers unfamiliar with this might skip the next paragraph and wait until we have introduced our own pictorial formalism for a detailed explanation).

(1.13). It is standard (Green [19]) that if A is an algebra and e an idempotent in A then there is an exact functor on the category of left A modules (in this paper *module* will mean left module unless otherwise stated):

$$G' : A - \text{mod} \to eAe - \text{mod}$$
$$G' : \mathcal{M} \mapsto e\mathcal{M}$$

(see [19, 32] for the morphism map). Consider the case in which A is H_{n+N}^N and $e = \frac{Y_N}{[N]!}$ (for $[N]! \neq 0$). Proposition 2 says that we may extend the functor trivially onto $H_n^N - \text{mod}$. This functor takes an irreducible representation to an irreducible representation (or zero), and is surjective. There is a right (but not left) inverse map

$$F' : eAe - \text{mod} \to A - \text{mod}$$
$$F' : \mathcal{N} \mapsto Ae \otimes_{eAe} \mathcal{N}$$

so that

$$F'G'(A) = Ae \otimes_{eAe} eA = Ae \otimes_{eAe} A \tag{7}$$

and

$$G'F'(\mathcal{N}) = eAe \otimes_{eAe} \mathcal{N} \cong \mathcal{N}. \tag{8}$$

(1.14). Now let

$$\Delta = \{\mathcal{N}_1, \mathcal{N}_2, \ldots\}$$

be a complete set of inequivalent simple modules of eAe. Then by equation (8)

$$F'(\mathcal{N}_1), F'(\mathcal{N}_2), \ldots$$

are distinct A modules. Suppose V is a proper submodule of $F'(\mathcal{N}_i)$. Then $eV = 0$ (for suppose $eV \neq 0$, then $G'(V) = eV \subseteq \mathcal{N}_i$, and thus in fact $eV \cong \mathcal{N}_i$, since the latter is a *simple* eAe-module, whereupon $V \supseteq AeV = F'(\mathcal{N}_i)$—a contradiction). Now let V_i be the sum of all submodules V with $eV = 0$. Then this is the unique maximal proper submodule of $F'(\mathcal{N}_i)$. Thus

$$F'(\mathcal{N}_1)/V_1, F'(\mathcal{N}_2)/V_2, \ldots$$

are inequivalent simple modules of A. If a simple module M has no equivalent in this list then $eM = 0$, so M is also an A/AeA module.

A useful example if this set-up is the well known result:

Proposition 3. For A, a k-algebra, let $e \in A$ be a primitive indempotent. Then the left ideal Ae has unique maximal proper submodule.

Proof. Primitivity implies $eAe = ke \cong k$ as a k-algebra. Thus $\mathcal{N}_1 = ke$ is (the only) simple ke-module, and so $F'(ke) = Ae$ has unique maximal proper submodule. □

To summarize: the irreducible representations $\mathcal{R}$ of $A -$ mod are in correspondence with those of $eAe -$ mod, except that those also in $A/AeA -$ mod (i.e. those for which $\mathcal{R}(e) = 0$, which will be taken to zero by G') do not have a correspondent in eAe and must be discovered separately.

For example, in our case, if we know the representations and morphisms (intertwiners) of H_n^N (and hence of $Y_N H_{n+N}^N Y_N$) we know those of H_{n+N}^N, except for those coming from $H_{n+N}^N / H_{n+N}^N Y_N H_{n+N}^N$, which we must determine by a separate calculation. But this quotient is just H_{n+N}^{N-1}, so again these may be determined by iteration (this time on N with base $H_{n+N}^1 \cong \mathbb{C}$). Indeed given proposition 2 we can move straight to a fundamental result on H_n^N:

Proposition 4. For $q, [N]! \neq 0$, irreducible representations of H_n^N are indexed by $\Lambda^+(n, N)$.

Outline proof. (We will complete this proof in section 3.3). By induction on n and N. For each n and N define f to be the isomorphism functor corresponding to proposition 2, and $F = F' f^{-1}$ and $G = fG'$. Now suppose the proposition true at level H_{n-N}^N and level H_n^{N-1} (of n and N respectively). Then all the irreducibles indexed by Young diagrams with less than N rows come from H_n^{N-1}, and the image of $\lambda \in \Gamma^+(n-N, N)$ under F is $\lambda + (1, 1, \ldots, 1)$. □

This is, as it were, a profoundly *mathematical* construction. One striking thing about it, therefore, is the fact that the map F preserves statistical mechanical observables in the following sense. The spectrum of a physical transfer matrix or Hamiltonian as in equation (6) breaks up into parts from distinct irreducible representations of $H_n(q)$ contained within it. Now, appropriately treated eigenvalues can be collected from a set of increasing values of n to form a converging sequence approaching a large n limit *observable*—but there is essentially only one way of doing this [8]. For example, for each n the free energy comes from the largest eigenvalue of the transfer matrix, and it is easy to figure out which irreducible representation gives this [34]. The functor F which takes us from one n to another correctly picks out the appropriate representation each time! There is also evidence that it maps spin–spin correlations to spin–spin correlations, and so on. A partial explanation is given in [32], but this deserves full generalization. In the present paper we deal with the mathematical side of this issue.

Another way of looking at this is through the q-Schur–Weyl duality $H_n^N \cong \mathrm{End}_{U_q\,\mathrm{sl}(N)}(V_N^{\otimes n})$ [33], which implies that for each n the index set for irreducibles of H_n^N maps injectively into the index set for irreducibles of $U_q\,\mathrm{sl}(N)$. Since this set is independent of n it provides a formal link between those irreducibles of $H_{n_1}^N$ and $H_{n_2}^N$ (say) with the same $U_q\,\mathrm{sl}(N)$ index. We will see in section 3 that the F and $U_q\,\mathrm{sl}(N)$ links coincide.

Note that Y_N is central in H_N^N. On the dual $U_q\,\mathrm{sl}(N)$ side it is a projector from $V_N^{\otimes N}$ to the trivial representation $V_{(1^N)}$. Proposition 2 may be verified in these terms, *given* the duality, by observing that $V_N^{\otimes n} \cong V_{(1^N)} \otimes V_N^{\otimes n}$. We will include a direct proof of the proposition, however, since this is self-contained and *very* much shorter than the proof of duality!

The functors F will also allow us to explicitly construct representations of any $H_n(q)$ by iterating from the known representations of $H_q(q) \cong \mathbb{C}$.

(1.15). Let A be an algebra, $M \in A-\mathrm{mod}$ and $\mathcal{S} \subset A-\mathrm{mod}$. We say that M has an $\mathcal{S}$-filtration if it has a finite series of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ such that for each $i = 1, 2, \ldots, l$ there is some $N \in \mathcal{S}$ such that $M_i/M_{i-1} \cong N$. Then N is called an $\mathcal{S}$-filtration factor of M. If the number of times N occurs in a given series for M is independent of the choice of series, then it is called the filtration multiplicity of N in M. For example, if $\mathcal{S}$ is a complete set of inequivalent simple modules then every finite dimensional module has an $\mathcal{S}$-filtration, and well defined filtration multiplicities (in this case called composition multiplicities) with respect to $\mathcal{S}$.

1.3. Physics motivation and interpretation

Physically, while proposition 4 is well known for generic q [22], it is even more useful for q roots of unity (or at least those with $[N]! \neq 0$, i.e. $q^{N+1} = 1$ and higher roots) since, while it is known that the size of irreducible representations generally gets smaller at roots of unity [33] we learn here that the number of distinct ones remains fixed. The only consistent explanation of this in a Hamiltonian representation $\mathcal{R}$ of dimension independent of q is that the *multiplicities* of irreducibles (and hence of Hamiltonian eigenvalues) increases at such a q, i.e. we have energy level crossings! The manner in which the functor F maps from H_n^N (via proposition 2) to H_{n+N}^N tells us that once such a level crossing occurs at some level n it will be present at all higher levels $n_l = n + lN$ in the sequence. Thus it is not an accidental crossing, but a phenomenon which will survive to the thermodynamic limit.

In fact the spin-chain representation $\mathcal{R}_N$ has (independent of q) a filtration with factors the set of generically irreducible representations we will construct (this is standard, see [21]), thus if we can determine the morphisms and composition series of these representations in terms of irreducibles (dependent on q) we have substantial information on the q dependence of spectrum degeneracies.

1.4. Explicit applications: level crossings

Fix N (indeed it may be helpful to think concretely of $N = 3$), and recall that a partial classification of states of the basic $n = Nm + l$ site $U_q\,\mathrm{sl}(N)$ invariant spin chain (n large, $l \in \{0, 1, \ldots, N-1\}$) can be made in terms of partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{N-1})$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{N-1} \geqslant 0$ and $\sum_{i=1}^{N-1} \lambda_i$ is congruent to l mod N. We see from above that this corresponds to a classification for irreducible representations of $H_n^N(q)$ in the large n limit, stabilized by the F and G functors. That is, the *fibre* of representations labelled $(\lambda_1, \lambda_2, \ldots, \lambda_{N-1})$ is all those representations with the usual index of the form $(p+\lambda_1, p+\lambda_2, \ldots, p+\lambda_{N-1}, p)$, $p \in \mathbb{N}$ (for given p the action of F takes $(p+\lambda_1, p+\lambda_2, \ldots, p+\lambda_{N-1}, p)$ to $(p+1+\lambda_1, p+1+\lambda_2, \ldots, p+1+\lambda_{N-1}, p+1)$).

The multiplicity (at fixed momentum) of an eigenvalue of the Hamiltonian in equation (6) will be given by the multiplicity of the corresponding irreducible representation ($L(\mu)$, say) in $\mathcal{R}_N$. By duality this multiplicity equals the dimension of the corresponding *indecomposable* summand $T'(\mu)$ of $V_N^{\otimes n}$ as a $U_q\,\mathrm{sl}(N)$-module (the 'tilting module' [17]—here we use *primes* on modules to distinguish $U_q\,\mathrm{sl}(N)$-modules, and for these μ should be read as the *fibre* label). The dimension of $T'(\mu)$ may be determined from its Weyl module content (since the dimensions of Weyl modules are known [37]). On general grounds [16] the multiplicity of Weyl module $\Delta'(\lambda)$ in tilting module $T'(\mu)$ coincides with that of $H_n^N(q)$ Specht module S^λ (see later) in $H_n^N(q)$ indecomposable projective module $P(\mu)$. And *these*, finally, are multiplicities which can be directly determined from results in this paper!

For example, later, in equations (31) and (32), we will exhibit a level crossing at $n = 5$ between states of the U_q sl(3) Hamiltonian corresponding to the generic irreducible representations indexed by (3,1,1) and (4,1), as q passes through $[5] = 0$. The point is, because of the F functor acting on the morphism between these two modules, our $n = 5$ result tells us that the $\lambda = (2)$ and $\lambda = (4, 1)$ states of the basic $l = 2$ U_q sl(3) spin chain *always* produce a level crossing at $q^5 = 1$ $(q \neq 1)$, i.e. for $n = 3m + 2$, any m. (In fact it is implicit in Wenzl [45] and in [32] that there are $r = 5$ level crossings associated with $\lambda = (2)$ in this model. The identification of the sector responsible is new, however.) Altogether, the thermodynamic limit multiplicity of the part of the Hamiltonian spectrum labelled by (4,1) increases at $[5] = 0$ from its generic level of 24-fold degeneracy (recall [37] that the dimension of U sl(3) Weyl module $\Delta'(\lambda)$ is $\frac{1}{2}(\lambda_1 + 2)(\lambda_1 - \lambda_2 + 1)(\lambda_2 + 1)$) to 30-fold degeneracy (30 being the dimension of the corresponding tilting module, computed via $T'(4, 1) = \Delta'(4, 1) + \Delta'(2)$—see section 3.4).

Further, we will see that the $\lambda = (4, 1)$ and $\lambda = (5)$ states also have crossings (this is implicit in [32]), but the states of the $(4, 1)$ sector involved in these crossings are disjoint from the crossings of $(4, 1)$ and (2).

1.5. Overview

To derive these results we are unavoidably concerned with mathematical tools. We will go into details only when they are physically or otherwise intuitively helpful, or when conducive to organizing in a physical way. Recent papers in this area use crystal base and algebraic geometry [2, 29]. These are beautiful, but not relevant by the above criteria. We treat them as black boxes. We will, however, provide sufficient background to enable their *use* to find level crossings. A few remarks on feedback of our results into representation theory are placed in appendix B.

Lascoux *et al* [29] give an algorithm for determining simple composition factors of Specht modules. This is data we need (we will discuss the role of Specht modules), but since it is only an algorithm it is not so useful without some additional organizational control over the data it produces. We will see that proposition 2 provides this. Also, note our thermodynamic limit process works through tracking module *morphisms*. The decomposition matrices which encode all the composition factors do not provide this data. However, some of it can be recovered by comparing them with Gram matrix determinant calculations (section 4).

In the next section we prove proposition 2. In section 3 we use this to construct 'standard' modules of $H_n^N(q)$ and to determine their images under the F and G functors, and in section 4 we look at the simple submodules of these modules and associated level crossings.

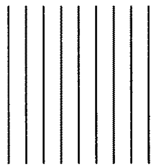
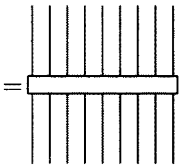
2. Diagrammatic proof of proposition 2

Having glimpsed the utility of proposition 2 we will now prove it. To avoid the opacity and tedium of an 'index chasing' proof (cf [32] for example) we introduce some diagrams, making crude use of the Hecke algebra's properties as a braid-group quotient. The basic idea is to help locate elements and subalgebras 'spatially' with respect to the 'strings' of the braid (the notion of i in g_i as a spatial coordinate comes directly from the role of g_i

in the physical transfer matrix [32]). We do not, in fact, make much use of these *as* braid diagrams, i.e. we do not often use the braid relations explicitly; however, our diagrams clarify situations in which two elements commute by virtue of being realized on disjoint subsets of strings.

The key for our diagrams is as follows.

(2.16). For given N the algebra H_n^N itself is represented as a blank rectangle across all the n strings of the corresponding braid group identity element. For example, with $n = 9$:

$1 =$  $H_n^N =$  .

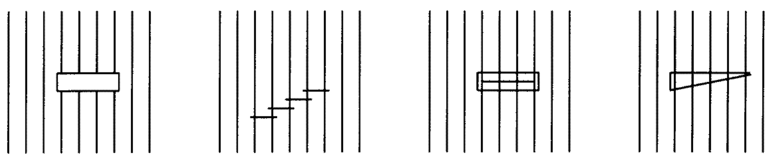
Note that by taking N sufficiently large we can make $H_9^N = H_9$, and so on. We will use the same picture when working with $\mathbb{H}_n$, since these cases may always be distinguished by context.

(2.17). For $m < n$ then H_m^N may be realized as a subalgebra of H_n^N in various ways, such as by acting on some subset of m of the n strings (the picture below shows a 'translated' embedding of $H_4^N \subset H_9^N$).

(2.18). Defining

$$g_{i,j} := g_i g_{i-1} \ldots g_{j+1} g_j \qquad (i > j)$$

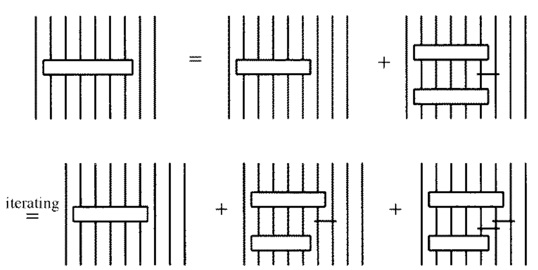
we will adopt the symbol scheme illustrated by the following examples:



'translated' H_4^N $\quad$ $g_{6,3} = g_6 g_5 g_4 g_3$ $\quad$ $Y_4^{(3)}$ $\quad$ $L_5^{(2)}$

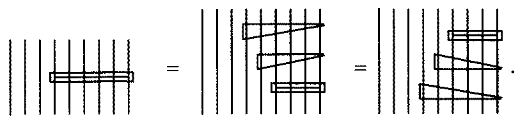
Note that we avoid the usual braid-crossing picture of g_i for the time being, as this can be distracting. We *will* use it later. Our point here is to view the various objects as 'beads' threaded on two or more strings, emphasizing which commute and which do not (g_i and g_{i+1} do not). We work as far as possible in $\mathbb{H}_n$ where we do not have access to g_i^{-1}, so this picture is not ambiguous. Note also that action from the left is depicted as action from the *top*, that is, while words are read from left to right, pictures are read *and composed* from top to bottom.

For example, the readily proven isomorphism of $\mathbb{Z}[q]$-modules $\mathbb{H}_{m+1} \cong \mathbb{H}_m \oplus \mathbb{H}_m g_m \mathbb{H}_m$ gives



(9)

(NB the middle term on the lower line simplifies), while equation (5) iterates to



(10)

(2.19). Here the last picture reminds us of the obvious generator *order* reversal symmetry $Y_N = Y_N^T$. This is time reversal symmetry in the statistical mechanical setting [32]. We have taken care in our choice of diagram shapes to respect the 'time' and 'space' symmetries of objects, thus Y_N is both top to bottom and left to right reflection symmetric (!), but L_N is neither (a right-angle triangle with the right angle in the bottom left-hand corner is thus $(L_N^{(k)})^T$ for some k and N).

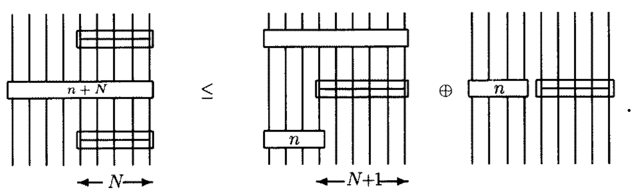
Noting the obvious inner automorphism of H_{n+N} taking Y_N to $Y_N^{(n)}$, we see that to prove proposition 2 it is enough to show

Lemma 1.

$$Y_N^{(n)}\mathbb{H}_{n+N}Y_N^{(n)} \leqslant \mathbb{H}_{n+N}Y_{N+1}^{(n-1)}\mathbb{H}_n \oplus Y_N^{(n)}\mathbb{H}_n \tag{11}$$

is an inclusion of $\mathbb{H}_n$-bimodules.

(Proposition 2 follows on passing to the field $\mathbb{C}$, since the first term on the right-hand side vanishes in H_{n+N}^N and there is an obvious morphism from the second term onto H_n^N when $[N]! \neq 0$.) Lemma 1 looks complicated, but may be illustrated by the following 'diagrammatic' proposition on $\mathbb{Z}[q]$-modules



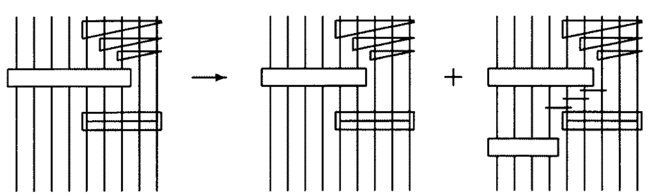
(12)

Proof. We claim that for $n - N < m \leqslant n$ there is a $\mathbb{Z}[q]$-module inclusion

$$(L_N^{(n)}L_{N-1}^{(n+1)}\dots L_{n-m+1}^{(m+N-1)})\mathbb{H}_{m+N}Y_N^{(n)} \hookrightarrow \mathbb{H}_{n+N}Y_{N+1}^{(n-1)}\mathbb{H}_n \oplus Y_N^{(n)}\mathbb{H}_n \tag{13}$$

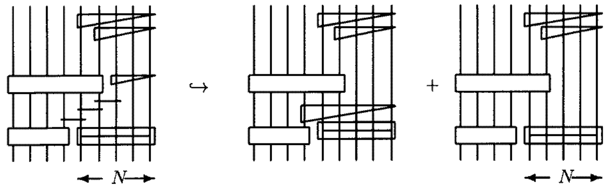
(i.e. into the right-hand side of equation (11)), and prove this by induction on m. The last case $(m = n)$ establishes our lemma.

As a base, the claim is true for $m = n - N + 1$ by suitably applying the top line of equation (9) to $\mathbb{H}_{n+1}$ in the left-hand side of equation (13). Suppose it is true at level $m = k - 1$. Then at level $m = k$ consider the inclusion, derived from another iterate of equation (9), illustrated below



We need to show that the left-hand side here maps into the right-hand side of equation (13). Our inductive assumption is that the first term on the right-hand side here does so, since

$L_{n-k+1}^{(k+N-1)} Y_N^{(n)} = [n-k+1] Y_N^{(n)}$, thus we have only to show that the second term does so. Using the definition of L_N and $g_i \mathbb{H}_n \subset \mathbb{H}_n (i < n)$ the second term obeys:



Using $L_{N+1} Y_N^{(1)} = Y_{N+1}$ (equation (5)) again the first term on the right here is manifestly contained in the first term in the right-hand side of equation (13) (consider it in the form of equation (12)), and the second term maps into the right-hand side of equation (12) by the inductive assumption. □

Now pass to any field in which $[N]!$ is invertible and the composite map to $Y_N^{(n)} H_n^N \cong H_n^N$ (as H_n^N-bimodules) is obviously surjective.

3. Applications of proposition 2

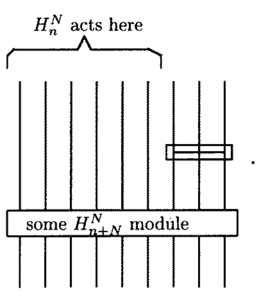
(3.20). Now apply the programme outlined in section 1.2, composing the functors F' and G' with the isomorphism of proposition 2 for each n and N. As above, it will be convenient to take $e = \frac{Y_N^{(n)}}{[N]!}$. From now on we will also show the levels n and N explicitly, hence the functors

$$H_n^N - \text{mod} \overset{F_n^N}{\to} H_{n+N}^N - \text{mod} \overset{G_n^N}{\to} H_n^N - \text{mod}$$

have object maps

$$F_n^N : M \mapsto H_{n+N}^N Y_N^{(n)} \otimes_{H_n^N} M \qquad \text{and} \qquad G_n^N : L \mapsto Y_N^{(n)} L.$$

The latter is illustrated by the following picture, in which an arbitrary H_{n+N}^N module is mapped to an $H_n^N Y_N^{(n)}$ module which is trivially also a left H_n^N module, since the $Y_N^{(n)}$ (here specifically $Y_3^{(6)}$) clearly commutes with the indicated action of H_n^N (here H_6^3) from the top



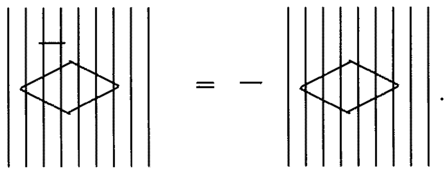
We next introduce and discuss the fate of the most important modules of $H_n(q)$ under these functors. Since the functors map between H_n^N and H_{n+N}^N, not $H_n(q)$ and $H_{n+N}(q)$, we need to be able to move between categories $H_n - \text{mod}$ and $H_n^N - \text{mod}$. To do this we note that H_n^N modules *restrict* to $H_n(q)$ modules which are identical to them as vector spaces, while $H_n(q)$ modules induce H_n^N modules which are not in general identical. In the next section we establish a simple condition under which they *are* identical.

3.1. Permutation modules of $H_n(q)$

(3.21). The image of Y_N under the $H_n(q)$ algebra involution Γ given by

$$\Gamma : g_i \mapsto -1 - g_i + q \tag{14}$$

is $X_N \in \mathbb{H}_N$, depicted by a diamond. The example below illustrates the identity $g_3 X_6^{(1)} = -X_6^{(1)}$:

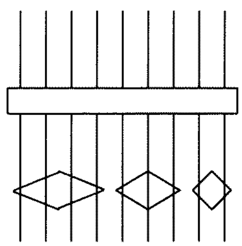


(3.22). For $\lambda \in \Lambda(n, N)$ define $X^\lambda, Y^\lambda \in \mathbb{H}_n$ by

$$X^\lambda = \prod_i X_{\lambda_i}^{(\sum_{j=1}^{i-1} \lambda_j)} \qquad Y^\lambda = \prod_i Y_{\lambda_i}^{(\sum_{j=1}^{i-1} \lambda_j)}$$

i.e. one factor X_{λ_i} (resp. Y_{λ_i}) for each row of the Young diagram of λ.

(3.23). For $\lambda \vdash n$ the $H_n(q)$ *permutation module* P^λ is given by $P^\lambda = H_n X^\lambda$ [21, 24], i.e. it takes the form



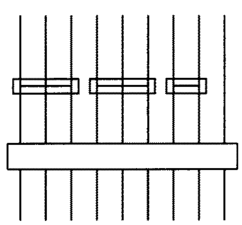
(15)

(this example is P^λ for $\lambda = (4, 3, 2)$, that is, $P^{(4,3,2)} = H_9 X_4 X_3^{(4)} X_2^{(7)}$). Note that $H_9 X_3 X_4^{(3)} X_2^{(7)}$ and other such rearrangements are isomorphic as left modules (at least provided $q \neq 0$). This P^λ specializes at $q = 1$ to the S_n induced module whose irreducible decomposition begins

$$(4) \otimes (3) \otimes (2) = (4, 3, 2) \oplus \dots \tag{16}$$

where the rest of the summands can be determined using the Littlewood–Richardson rules [24, 21].

(3.24). The right $H_n(q)$ module $Q_H^\lambda := Y^{\lambda'} H_n$. In our case:

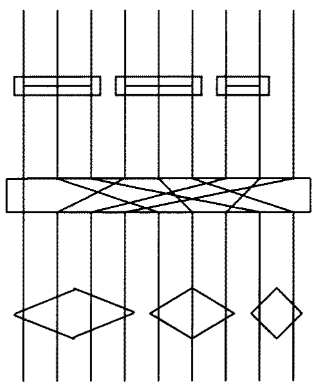


(note that $\lambda' = (3, 3, 2, 1)$ so we have $Q^\lambda_H = Y_3 Y^{(3)}_3 Y^{(6)}_2 Y^{(8)}_1 H_9$ with $Y_1 = 1$). This specializes at $q = 1$ to the S_n-induced right module

$$(1^3) \otimes (1^3) \otimes (1^2) \otimes (1) = (4, 3, 2) \oplus \ldots$$

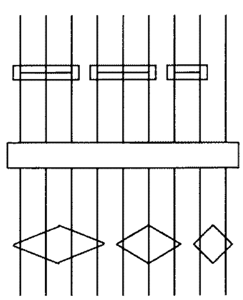
(an otherwise different set of summands to equation (16)).

(3.25). For $\lambda \vdash n$ consider an element $\sigma^\lambda_g \in \mathfrak{B}_n$ such that, when the strings are partitioned in the natural way by λ' at the top and by λ at the bottom, no two strings in a given part at the top (resp. bottom) either cross each other or travel to the same part at the bottom (resp. top). This construction is unique, since it forces the ith string from the jth part at the top to travel down to be the jth string in the ith part at the bottom. Put $E^\lambda_g := Y^{\lambda'} \sigma^\lambda_g X^\lambda$, for example



where each crossing should be interpreted as a g_i (NB putting a g_i^{-1}, or a linear combination other than 1, at each crossing only changes the element by a scalar in this environment). Note that $\mathfrak{C}_{\sigma^\lambda_g}(E^\lambda_g) = 1$ by construction. Hence

Proposition 5. The vector space $Q^\lambda_H \otimes_{H_n} P^\lambda = \mathbb{C}E^\lambda_g$, i.e.



(17)

is one dimensional.

NB The tensor product is strictly superfluous here. We leave it in as a guide to the eye.

(3.26). Let $\mathcal{A}$ be a k-algebra. A *primitive* element $x \in \mathcal{A} \setminus \{0\}$ is such that $x\mathcal{A}x \subseteq kx$. A *pre-idempotent* is a primitive such that $xx \neq 0$.

Proposition 6. E^λ_g is primitive in $H_n(q)$ (any q), and $I^\lambda := (Y^{\lambda'} \sigma^\lambda_g X^\lambda (\sigma^\lambda_g)^T Y^{\lambda'})$ is pre-idempotent for q generic.

Proof. Using proposition 5

$$(Y^{\lambda'}\sigma_g^\lambda X^\lambda)H_n(Y^{\lambda'}\sigma_g^\lambda X^\lambda) = Y^{\lambda'}(\sigma_g^\lambda X^\lambda H_n Y^{\lambda'}\sigma_g^\lambda)X^\lambda = kY^{\lambda'}\sigma_g^\lambda X^\lambda.$$

Further, recall that if y is any element of a semi-simple algebra then there exists an element a such that $yay = y$. Now suppose $I^\lambda I^\lambda = 0$ for some q such that $[\lambda']!$ is invertible. Then for any a we can use proposition 5 again to get

$$\begin{aligned}
I^\lambda a I^\lambda &= (Y^{\lambda'}\sigma_g^\lambda X^\lambda(\sigma_g^\lambda)^T Y^{\lambda'})a(Y^{\lambda'}\sigma_g^\lambda X^\lambda(\sigma_g^\lambda)^T Y^{\lambda'}) \\
&= Y^{\lambda'}\sigma_g^\lambda X^\lambda(\sigma_g^\lambda)^T(Y^{\lambda'}aY^{\lambda'}\sigma_g^\lambda X^\lambda)(\sigma_g^\lambda)^T Y^{\lambda'} \\
&= c_a Y^{\lambda'}\sigma_g^\lambda X^\lambda(\sigma_g^\lambda)^T(Y^{\lambda'}\sigma_g^\lambda X^\lambda)(\sigma_g^\lambda)^T Y^{\lambda'} \\
&= \frac{c_a}{[\lambda']!}(Y^{\lambda'}\sigma_g^\lambda X^\lambda(\sigma_g^\lambda)^T Y^{\lambda'})(Y^{\lambda'}\sigma_g^\lambda X^\lambda(\sigma_g^\lambda)^T Y^{\lambda'}) \\
&= \frac{c_a}{[\lambda']!} I^\lambda I^\lambda = 0
\end{aligned}$$

(some $c_a \in \mathbb{C}$ [13]) contradicting semi-simplicity. □

(3.27). The dominance order [30] is a partial order on partitions of n given by

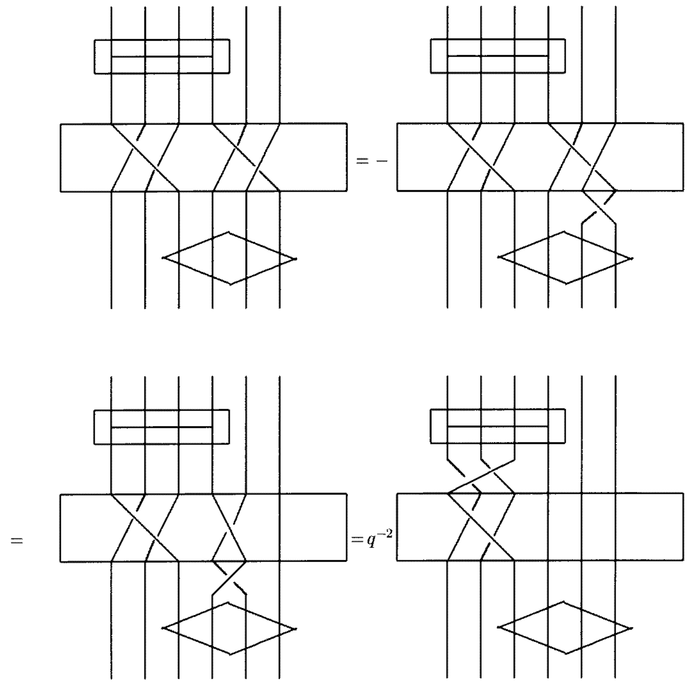
$$\lambda \trianglerighteq \mu \qquad \text{if } \sum_{i=1}^m \lambda_i \geqslant \sum_{i=1}^m \mu_i \;\; \forall m.$$

For example

$$(6) \trianglerighteq (5,1) \trianglerighteq (4,2) \begin{matrix} \trianglerighteq (3,3) \trianglerighteq \\ \trianglerighteq (4,1^2) \trianglerighteq \end{matrix} (3,2,1) \begin{matrix} \trianglerighteq (2^3) \trianglerighteq \\ \trianglerighteq (3,1^3) \trianglerighteq \end{matrix} (2,2,1^2) \trianglerighteq (2,1^4) \trianglerighteq (1^6).$$

Proposition 7. The vector space $Q_H^\lambda \otimes_{H_n} P^\nu = 0$ unless $\lambda \trianglerighteq \nu$.

Proof. Consider $Y^{\lambda'}bX^\nu$, $b \in \mathfrak{B}_n$. Suppose two strings from some $Y_{\lambda'_i}$ at the top of the braid picture eventually reach a given X_{ν_j} at the bottom (as must happen unless $\lambda \trianglerighteq \nu$, by an elementary sorting argument). Then there is always a set of moves using the braid relations and $qY = g_iY$ and $g_iX = -X$ (proposition 1), such as those illustrated below, to bring these strings 'parallel'. For example, focusing on a particular Y_N and X_M we might have



(18)

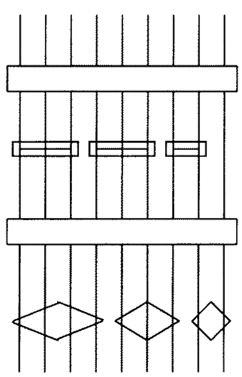
whereupon the diagram vanishes using $Y_2 X_2 = 0$ (equation (3)). □

This is an extremely useful observation. It means that any left $H_n(q)$-module P^λ, or submodule thereof, is also an H_n^N module (i.e. induces an H_n^N module identical to it as a vector space) provided that $\lambda_1' \leqslant N$. That is, in such a case $Y_{N+1} H_n P^\lambda = 0$.

We are now in a position to construct the generically irreducible modules of $H_n(q)$.

3.2. Specht modules of $H_n(q)$ (standard modules of H_n^N)

(3.28). The $H_n(q)$ left *Specht module* S^λ [45] (see also [14, 28, 15]) may be written $H_n I^\lambda \cong H_n Q_H^\lambda \otimes_{H_n} P^\lambda$, for example:


. (19)

Note that this construction is well defined for any value of q. By proposition 5 such modules are irreducible when $H_n(q)$ is semi-simple. The particular one illustrated above specializes at $q = 1$ to the irreducible representation usually labelled $\lambda = (4, 3, 2)$ [24]. Our labelling is consistent with the way irreducibles are normally labelled in the $q = 1$ case.

The Specht module S^λ is a left H_n module, not an H_n^N module, so we cannot talk directly of $F_n^N(S^\lambda)$. However, S^λ obeys $Y_{\lambda'_1+1}S^\lambda = 0$ by our argument above, so it will induce an $H_n^{\lambda'_1}$ module identical to it as a vector space (and of course this $H_n^{\lambda'_1}$ module will be restricted to an 'identical' H_n module). By taking care we can thus move backward and forward between H_n and H_n^N and apply the functors to these modules. Where sensible (i.e. for $N = \lambda'_1$ as above, and in fact similarly for $N > \lambda'_1$) we will not explicitly distinguish between S^λ as an H_n module and S^λ as an H_n^N module.

3.3. *Images of Specht modules under the factors F_n^N and G_n^N*

The images of S^λ under the appropriate functors F_n^N and G_n^N are as follows. For a given N then $\lambda = (\lambda_1, \dots, \lambda_N)$ (some of these may be zero), and

$$F_n^N(S^\lambda) \cong S^{\lambda_+} \tag{20}$$

where $\lambda_+ = (\lambda_1 + 1, \dots, \lambda_N + 1)$. For the domain of G_n^N then either $\lambda_N \geqslant 1$ and the obvious reverse map pertains:

$$G_n^N(S^\lambda) \cong S^{\lambda_-} \tag{21}$$

where $\lambda_- = (\lambda_1 - 1, \dots, \lambda_N - 1)$, or $\lambda_N = 0$ and $G_n^N(S^\lambda) = 0$ by (3.28). These maps may be established as follows.

First, the left module $F_n^N(S^\lambda)$ is given by

$$F_n^N(S^\lambda) = H_{n+N}^N Y_N^{(n)} \otimes_{H_n^N} (Q_H^\lambda \otimes_{H_n^N} P^\lambda).$$

To classify this module at level $n + N$ it is convenient to define $g^\lambda \in \mathbb{H}_n$ by

$$g^\lambda = \prod_{i=1}^{\lambda'_1} \left(g_{n+i-1, i+\sum_{j=1}^{i} \lambda_i} \right)$$

(this looks complicated, but is very simple in the pictorial version as we will see in the next figure) and to note that

$$H_n^N Q_H^\lambda \otimes P^\lambda g^\lambda X^{\lambda_+} \cong H_n^N Q_H^\lambda \otimes P^\lambda \tag{22}$$

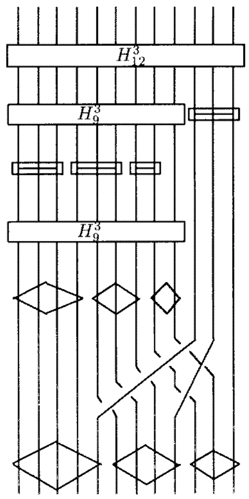
as an H_n^N module. To see this look at the bottom three layers of the figure below—three strings pulled off to the right play no role, and

$$X^\lambda g^\lambda X^{\lambda_+} = [\lambda]! g^\lambda X^{\lambda_+} \tag{23}$$

since $X_N X_{N+1} = [N]! X_{N+1}$. Working with this we end up with a module isomorphism

$$F_n^N(S^\lambda) \cong H_{n+N}^N \underbrace{\underbrace{Y_N^{(n)} Q_H^\lambda}_{Y^{\lambda_+} H_n^N} \otimes g^\lambda X^{\lambda_+}}_{\overset{\text{prop.5}}{=} Q_H^{\lambda_+} \otimes P^{\lambda_+}} \cong S^{\lambda_+}$$

which, for $N = 3$ and $n = 9$ looks like



$$. \tag{24}$$

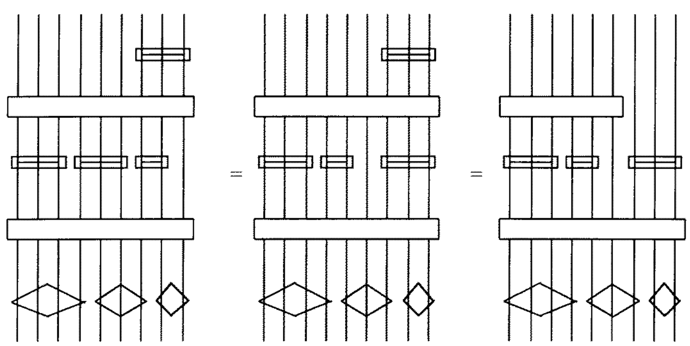
The left module $G^N_n(S^\lambda)$ is given by

$$G^N_n(S^\lambda) = Y_N^{(n-N)} H^N_n Q^\lambda_H \otimes P^\lambda = \begin{cases} 0 & \text{if } \lambda'_1 < N \\ \underbrace{Y_N^{(n-N)} H^N_n Y_N^{(n-N)}}_{\cong H^N_{n-N}} Q^{\lambda_-}_H \otimes P^\lambda & \text{if } \lambda'_1 = N \end{cases} \tag{25}$$

and the latter case becomes

$$H^N_{n-N} Q^{\lambda_-}_H \otimes H_n X^{\lambda_-} g^{\lambda_-} X^\lambda \cong S^{\lambda_-}$$

by a move analogous to equation (22). The latter case is illustrated by



$$\tag{26}$$

where the action of H^N_{n-N} is at the top left. □

The modules $\{S^\lambda | \lambda \vdash n\}$ form a complete set of irreducibles for generic q, and from propositions 2 and 4 we know that their maximal semi-simple quotients are simple and form a (possibly over-) complete set of irreducibles for *any* q.

3.4. Induction and restriction

Since Specht modules are well defined for any q we may deduce partial restriction rules for the inclusion $H_{n-1} \hookrightarrow H_n$ from the classical $q = 1$ case. These are:

$$\mathrm{Res}^{H_n}_{H_{n-1}}(S^\lambda) \cong \underset{\mu=\lambda-\Box}{+} S^\mu \tag{27}$$

as vector spaces, where $\mu = \lambda - \Box$ denotes $\mu \vdash n-1$ obtained by subtracting a box from the Young diagram λ and the sum is necessarily a direct sum of H_{n-1} modules (in characteristic 0) *unless* q is a root of unity.

This formula, together with the linkage principal [26, ch 6] and proposition 2 is enough to determine all composition multiplicities for $N = 2, 3$, and most (possibly all [35]) for general N. For example, the fibre of $N = 3$ Specht modules labelled by $\lambda = (2)$ (i.e. $S^{(2)}$, $S^{(3,1,1)}$, $S^{(4,2,2)}$ and so on) are easily shown to be projective in case $[5] = 0$. Restriction takes projective to projective, so applying three successive restrictions as above to (say) indecomposable projective $P(4, 2, 2)$ produces a new projective with direct summand $P(4, 1)$, itself a non-direct sum of the Specht modules for the fibres $(4, 1)$ and (2) (see section 4). Other projectives are determined similarly by iteration on the usual dominance order [23] on the fibre labels.

Mathematically oriented readers will also note that we may deduce that each algebra $H^N_n(q)$ for which the Specht module corresponding to fibre $\lambda = (0)$ is projective, is a quasi-hereditary algebra (cf [16]).

4. Inner products and submodules

Although Specht modules are generically irreducible they are not in general irreducible at roots of unity. Suppose we know the structure of $H_m(q)$ for some q at all $m < n$. In particular suppose S^λ develops a submodule at this q, i.e. for at least one μ

$$\begin{array}{ccccccccc}
 & & & 0 & & 0 & & & \\
 & & & & \searrow \quad \nearrow & & & & \\
 & & & & S^\mu/V & & & & \\
 & & & \nearrow & & \searrow & & & \\
 & & S^\mu & & & & S^\lambda & & \\
 & \nearrow & & & & & & \searrow & \\
 V & & & & & & & & L \\
 \nearrow & & & & & & & & \searrow \\
0 & & & & & & & & 0
\end{array} \tag{28}$$

(diagonal sequences short exact). Then there are two cases to consider. Either $\mu'_1 = \lambda'_1$ and we already know about the composite

$$S^\mu \xrightarrow{\psi} S^\lambda$$

via $F^{\mu'_1}_{n-\mu'_1}$, or $\mu'_1 < \lambda'_1$ and ψ is 'new'. In the latter case we can hope to find it by looking at the Smith form of the Gram matrix [44, 25] for some inner product on S^λ.

(4.29). A Gram matrix for S^λ may be constructed as follows. Let A and B be two bases for S^λ. Each can be of the form

$$A \subset \{wE^\lambda_g | w \in \mathfrak{B}_n\}$$

so for $x \in A$ and $y \in B$ we have an inner product $\langle x|y\rangle$ via

$$x^T y = (X^\lambda (\sigma_g^\lambda)^T Y^{\lambda'} w_x^T)(w_y Y^{\lambda'} (\sigma_g^\lambda) X^\lambda) \stackrel{\text{defn}}{=} \langle x|y\rangle X^\lambda (\sigma_g^\lambda)^T Y^{\lambda'} \sigma_g^\lambda X^\lambda.$$

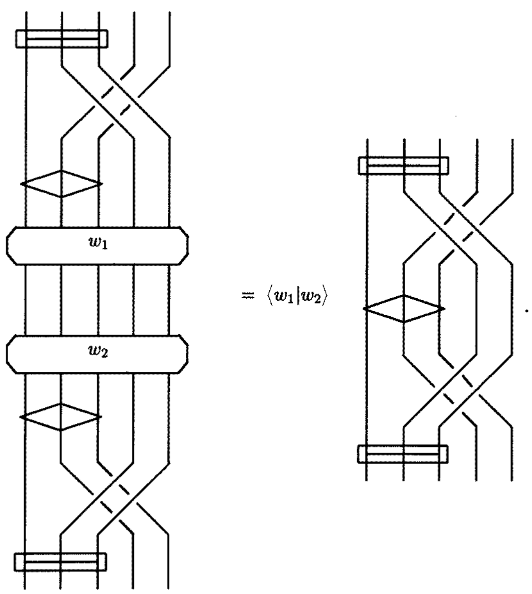
Then

$$C_{xy}^\lambda \stackrel{\text{defn}}{=} \langle x|y\rangle \in \mathbb{Z}[q]$$

gives the matrix elements of the Gram matrix C^λ (with respect to A, B).

If $V_{q_c}^\lambda$ is the maximal proper submodule of S^λ at $q = q_c$ then $\dim(V_{q_c}^\lambda)$ is the number of invariant factors of C^λ which vanish at $q = q_c$. Indeed, the number of invariant factors vanishing like each power of $(q - q_c)$ tells us the dimension of some subquotient of $V_{q_c}^\lambda$. Taken with our category theory data and Frobenius reciprocity [7] this kind of data is often enough to identify $V_{q_c}^\lambda$ (cf [31]). In any case, any vanishing invariant factor (hence vanishing $\det(C^\lambda)$) signals a level crossing of the Hamiltonian at $q = q_c$.

Note that Γ takes $E_g^\lambda \mapsto (E_g^{\lambda'})^T$. Thus $\det(C^\lambda)$ may similarly be computed via $(E_g^{\lambda'}) w_x^T w_y (E_g^{\lambda'})^T = D_{xy}^\lambda I^{\lambda'}$. Once again, our pictures simplify the issue. Let us illustrate with an example of the D^λ type.

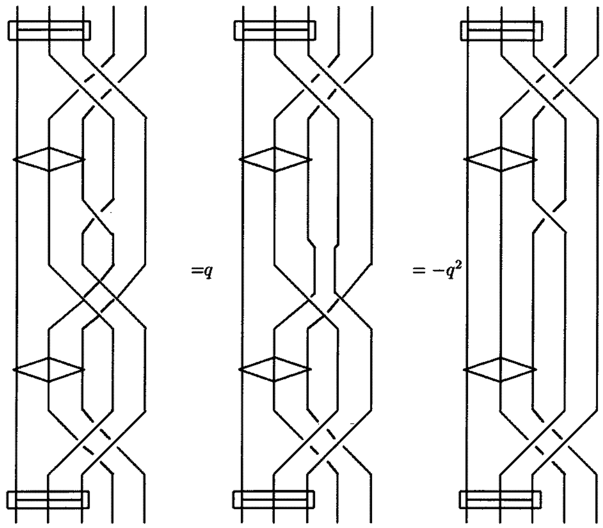


(29)

It is straightforward to construct a basis of S^λ. Consider the case $\lambda = (3, 1, 1)$. Let 12311 be the Yamanouchi symbol [6], and associate this to E_g^λ, then $g_3 E_g^\lambda = 12131$ (g_i interchanging the ith and $(i+1)$th letters), and so on until all the legal symbols are generated. A typical calculation is then

$$E_g^\lambda g_3 \cdot g_3 g_2 g_4 g_3 (E_g^\lambda)^T = \langle 12131|11123\rangle I^\lambda$$

which is illustrated by:



$$= q \qquad = -q^2 \qquad . \tag{30}$$

Here the first identity follows from the expansion of equation (3)

$$g_i^2 = q1 + (q-1)g_i$$

and $X_2Y_2 = 0$, and the second uses $g_iX_3 = -X_3$ and $g_iY_3 = qY_3$. Now in the present environment $X_4 = 0$, so from equation (35) of the appendix we have

$$X_3g_3X_3 = -q^3[3-1]!X_3.$$

Writing E^TijkE for $E_g^\lambda g_ig_jg_k(E_g^\lambda)^T$, the complete set of basis element products is

$$\begin{pmatrix} E^TE & E^T3E & E^T23E & E^T43E & E^T243E & E^T3243E \\ E^T3E & E^T33E & E^T323E & E^T343E & E^T3243E & E^T33243E \\ E^T32E & E^T323E & E^T3223E & E^T3243E & E^T32243E & E^T323243E \\ E^T34E & E^T343E & E^T3423E & E^T3443E & E^T34243E & E^T343243E \\ E^T324E & E^T3243E & E^T32423E & E^T32443E & E^T324243E & E^T3243243E \\ E^T3243E & E^T32433E & E^T324323E & E^T324343E & E^T3243243E & E^T32433243E \end{pmatrix}$$

We soon arrive at the Gram matrix (coefficients of, as it were, I^λ, in the above):

$$D^{(3,1,1)} = \begin{pmatrix} (q+1)(q^2+q+1) & q^3(q+1) & -q^3(q+1) & q^4(q+1) & -q^4(q+1) & 0 \\ q^3(q+1) & q(q+1)(q^3+q+1) & q^3(q+1) & q^5(q+1) & 0 & -q^5(q+1) \\ -q^3(q+1) & q^3(q+1) & q^2(q+1)(q^3+q^2+1) & 0 & q^6(q+1) & -q^6(q+1) \\ q^4(q+1) & q^5(q+1) & 0 & q^2(q+1)(q^4+q+1) & q^4(q+1) & q^5(q+1) \\ -q^4(q+1) & 0 & q^6(q+1) & q^4(q+1) & q^3(q+1)(q^4+q^2+1) & q^6(q+1) \\ 0 & -q^5(q+1) & -q^6(q+1) & q^5(q+1) & q^6(q+1) & q^4(q+1)(q^4+q^3+1) \end{pmatrix}$$

which has

$$\det(D^{(3,1,1)}) = q^{12}(q+1)^6(q^4+q^3+q^2+q+1)^3 = q^{12}[2]^6[5]^3. \tag{31}$$

In this case the invariant factors can be readily deduced, and we find that we are looking for a three-dimensional invariant subspace at $r = 5$. There is only one possibility for this submodule—the irreducible quotient of $S^{(4,1)}$ by $S^{(5)}$ (see [32]), i.e. the $r = 5$ module $S_5^{(4,1)}$ defined by the short exact sequence

$$0 \to S^{(5)} \to S^{(4,1)} \to S_5^{(4,1)} \to 0. \tag{32}$$

That is, we have a diagram as in equation (28) with $\mu = (4, 1)$ and $\lambda = (3, 1, 1)$. This gives the q-level crossing discussed in section 1.4.

A complete list of Gram matrix determinants up to $n = 4$ is given in appendix B (in all these cases the invariant factors are readily deduced). These determinants should be compared with the S_n results of James and Murphy [25]. Mathas and James [36] have recently given an algorithm for computing the determinant of the Gram matrix in general, which we also discuss in the appendix.

We have already seen how these results can be used to deduce all the relevant level crossings. It is reasonable to suppose that we can 'bootstrap' these results to higher n to derive *all* the Smith forms and all the corresponding crossings. This work is in progress.

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Appendix A. Useful identities

First

$$\begin{aligned} Y_{N+1} = \frac{Y_N}{[N]!} Y_{N+1} = \frac{Y_N}{[N]!} L_{N+1} Y_N = \frac{Y_N}{[N]!} (1 + g_N + g_{N-1} g_N + \cdots + g_1 g_2 \ldots g_N) Y_N \\ = \frac{Y_N}{[N]!} (1 + L_N g_N) Y_N = \frac{Y_N}{[N]!} (1 + [N] g_N) Y_N = Y_N + \frac{1}{[N-1]!} Y_N g_N Y_N \end{aligned} \tag{33}$$

so

$$Y_N g_N Y_N = [N-1]! (Y_{N+1} - Y_N). \tag{34}$$

Then applying Γ

$$X_N g_N X_N = q^N [N-1]! X_N - X_{N+1}. \tag{35}$$

Similarly

$$\begin{aligned} Y_N g_N g_{N+1} g_{N-1} g_N Y_N = [N-2]! (Y_{N+2} - (Y_{N+1} + g_{n+1} Y_{N+1} + Y_{N+1} g_{n+1} \\ + g_{n+1} Y_{N+1} g_{n+1}) + q Y_N + q g_{n+1} Y_N). \end{aligned} \tag{36}$$

Appendix B. Gram matrices and Smith forms

A recursive construction of the determinant $\det \mu$ of the Gram matrix for Specht module S^μ of dimension $\dim \mu$ (cf the symmetric group case [25], replacing use of the orthogonal form of Young [47] with that of Hoefsmit [22]) is as follows. Let I be the set of row positions of μ from which a box may be removed, and for $i \in I$ let μ^i denote the corresponding subdiagram. For $i \in I$ let J_i be the set of hook lengths of μ in the column above the removable box. Then

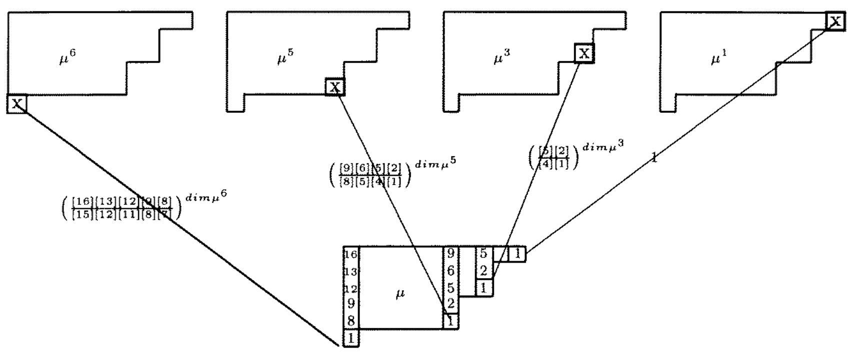
$$\det \mu = \prod_{i \in I} \det \mu^i \left(q^{x(\mu^i)} \prod_{j \in J_i} \frac{[j]}{[j-1]} \right)^{\dim \mu^i}$$

(note that the empty product $\prod_{j\in J_1}(anything) := 1$), where $x(\mu^i)$ is defined as follows.

(B.30). For $\mu^i \subset \mu$ as above the set $\mathfrak{y}(\mu^i)$ of standard (tableau) Yamanouchi symbols of μ^i maps into that of μ by $Y^i : y \mapsto yi$ (and $\cup_{i\in I} Y^i(\mathfrak{y}(\mu^i)) = \mathfrak{y}(\mu)$). For $y \in \mathfrak{y}(\mu)$ and y^0_μ the Bruhat lowest element (e.g. $y^0_\mu = 12311$ for $\mu = (3, 1, 1)$) let $l(y)$ be the minimum number of adjacent element transpositions to move from y^0_μ to y. Then $x(\mu^i) := l(Y^i(y^0_{\mu^i}))$.

The result is due to Mathas and James (the $q^{x(\mu^i)}$ part we conjecture from the results of several explicit calculations).

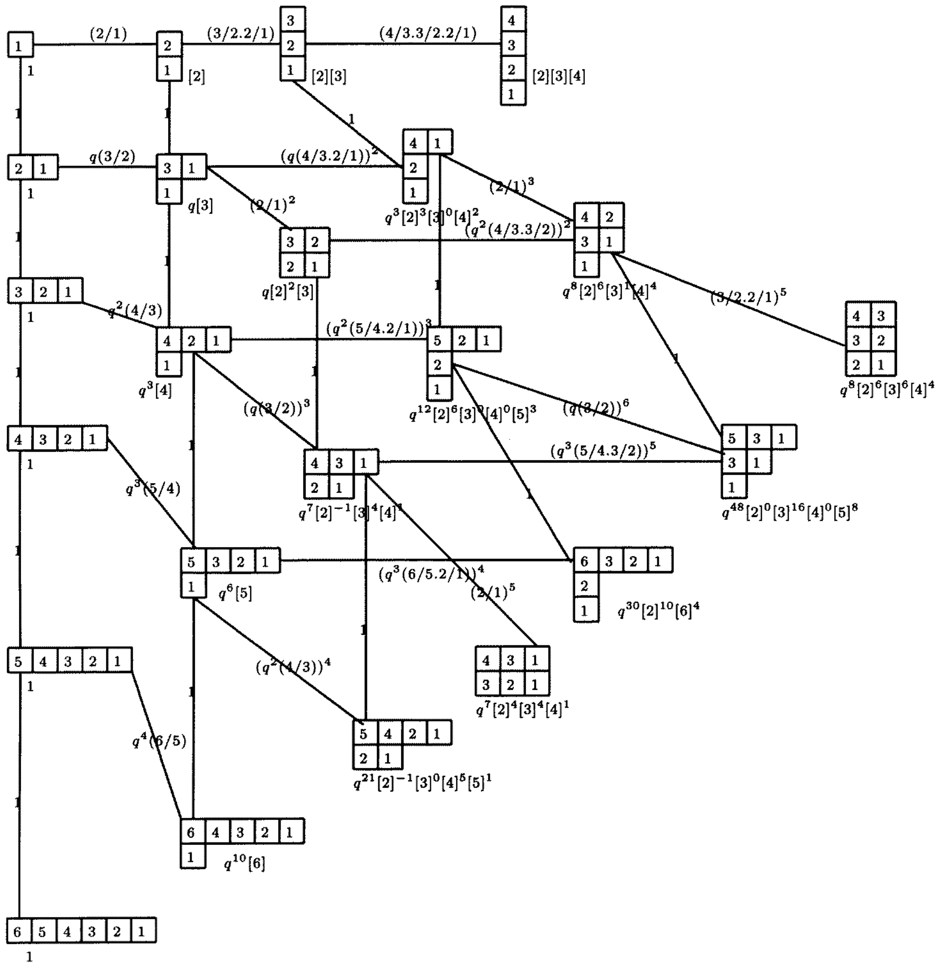
For example:



$$\det\mu = \det\mu^6\cdot\det\mu^5\cdot\det\mu^3\cdot\det\mu^1\left(q^{x(\mu^6)}\frac{[16][13]\ldots[8]}{[15][12]\ldots[7]}\right)^{\dim\mu^6}$$
$$\times\left(q^{x(\mu^5)}\frac{[9][6][5][2]}{[8][5][4][1]}\right)^{\dim\mu^5}\left(q^{x(\mu^3)}\frac{[5][2]}{[4][1]}\right)^{\dim\mu^3}.$$

(B.31). For A an algebra and $M \in A-$mod let $\mathrm{Top}(M)$ be the maximal semi-simple quotient of M. Let $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ be a sequence of submodules such that $M_i/M_{i+1} = \mathrm{Top}(M_i)$, then put $\mathrm{Top}^i(M) := \mathrm{Top}(M_i)$. In case this Top series exactly reverses the Socle series of [18] we will call $\mathrm{Top}^i(M)$ the ith Loewy layer of M.

We conjecture that the power of $(q-q_r)$ which divides $\det\mu$ is $\sum_l l\dim(\mathrm{Top}^l(S^\mu|_{q=q_r}))$. The conjecture is correct in the Temperley–Lieb ($N = 2$) sector [32, 46]. We have verified the conjecture by explicit calculation (as discussed in the text above) for $N = 3$ up to $n = 6$. The results are shown (under the diagram to which they correspond) in the following figure. We have included all hook lengths, and an indication of the 'hook ratio' factor associated by the determinant algorithm to each edge of the restriction graph, writing (5/4.2/1) for $\frac{[5][2]}{[4][1]}$, and so on. By this means the reader will readily confirm the agreement.



These results tie in with the conjecture of Lascoux *et al* [29] concerning Jantzen filtration (see also [36]). For example at $r = 4$, $n = 8$ we have the Loewy layer decomposition into irreducibles:

$$S^{(3,3,2)} = \begin{matrix} L(3,3,2) \\ L(4,3,1) \\ L(8) \end{matrix}.$$

The analysis is as follows. The three irreducibles shown are the composition factors of $S^{(3,3,2)}$ by an application of section 3.4, or by [2, 29]. Their dimensions are 1, 40 and 1 respectively (the first of these follows from [22] or [45], the last is obvious, and the second is thus forced). On the other hand

$$\det(3,3,2) = q^{l_{(3,3,2)}}[2]^{42}[3]^{21}[4]^{42}[5]^{21} \qquad \text{where } l_\lambda = \sum_{y \in \mathfrak{y}(\lambda)} l(y).$$

Thus the naive upper bound on the dimension of the maximal submodule at $r = 4$ is 42, and is not reached. The only possibility is that the Smith form of the Gram matrix has 40 invariant factors vanishing like [4] (corresponding to $L(4,3,1)$ at Loewy level 1) and one invariant factor vanishing like $[4]^2$ (corresponding to $L(8)$ at Loewy level 2), leaving one factor nonvanishing. The point is that the Lascoux *et al* [29] tables include what amounts to a formal parameter (q), and the power of this parameter agrees with the Loewy level (cf [46]).

We have checked several such cases. To illustrate the method, let us determine all H_n^3 Specht module morphisms for $r = 4, l = 2$ arising up to $n = 2.3 + 2 = 8$. Inspection of our determinant table at $n = 0$, $N + 2 = 2$ shows no μ such that $[4]|\det\mu \in K$, thus no morphisms coming from this level. At $n = 1$, $N + 2 = 5$, $\det(3,2) \sim [4]$ and $\det(2,2,1) \sim [4]^4$. The first of these gives the exact sequence

$$0 \to L(5) \to S^{(3,2)} \to L(3,2) \to 0$$

since there is only one candidate for a one-dimensional submodule. Noting hence that $\dim(L(3,2)) = 4$, the second determinant gives exact

$$0 \to L(3,2) \to S^{(2,2,1)} \to L(2,2,1) \to 0$$

(a simple check shows that $S^{(4,1)}$ is not involved). This is everything.

Passing to $n = 2.N + 2 = 8$ with the F functor these data become sequences:

$$0 \to S^{(6,1,1)} \to S^{(4,3,1)} \to L(4,3,1) \tag{37}$$

and

$$0 \to S^{(6,1,1)} \to S^{(4,3,1)} \to S^{(3,3,2)} \to L(3,3,2). \tag{38}$$

But in the first of these we lose exactness at $S^{(4,3,1)}$ in principle [19] (and in practice)—i.e. $\ker(p)$ is bigger than just $S^{(6,1,1)}$; and equation (38) is not exact at $S^{(4,3,1)}$ or $S^{(3,3,2)}$. However F takes projectives to projectives, so $S^{(3,3,2)}$ is projective.

From the $N = 2$ solution [32, p 174] we already know

$$0 \to L(8) \to S^{(7,1)} \to L(7,1) \to 0 \qquad \text{and} \qquad 0 \to L(7,1) \to S^{(4,4)} \to L(4,4) \to 0$$

and of course $\dim(L(8)) = 1$, so $\dim(L(7,1)) = 6$ and $\dim(L(4,4)) = 8$; while $\det(6,1,1) = q^{l_{(6,1,1)}}[2]^{21}[4]^6[8]^6$ gives

$$0 \to L(7,1) \to S^{(6,1,1)} \to L(6,1,1) \to 0$$

so $\dim(L(6,1,1)) = 15$. Lascoux *et al*'s [29] table tells us that $L(4,3,1)$, $L(8)$, $L(4,4)$, $L(7,1)$ appear in $S^{(4,3,1)}$ each with multiplicity 1. Our morphisms show $S^{(6,1,1)} \subset S^{(4,3,1)}$, thus $L(8)$, $L(7,1)$ remain. Since $\det(4,3,1) = q^{l_{(4,3,1)}}[2]^{-21}[3]^{64}[4]^{36}[6]^{27}$ and

$$36 - \dim(L(6,1,1)) - 2 \cdot \dim(L(7,1)) = 9$$

($L(7,1)$ is 'below' $L(6,1,1)$) we can locate these in the first Loewy layer. We have a block [13]

$$S^{(4,3,1)} = \begin{matrix} & L(4,3,1) & \\ L(8) & L(6,1,1) & L(4,4) \\ & L(7,1) & \end{matrix} \qquad S^{(6,1,1)} = \begin{matrix} L(6,1,1) \\ L(7,1) \end{matrix}$$

$$S^{(4,4)} = \begin{matrix} L(4,4) \\ L(7,1) \end{matrix} \qquad S^{(7,1)} = \begin{matrix} L(7,1) \\ L(8) \end{matrix}$$

and $S^{(8)} = L(8)$ (and $S^{(3,3,2)}$ as above). Again this 'agrees' with the Lascoux *et al* [29] q-depth data. A good guess for the form of a modest $n = 8$ projective module is thus

$$
\begin{array}{cccccc}
 & & & L(8) & & \\
 & & L(7,1) & L(4,3,1) & & \\
P(8) = & L(8) & L(3,3,2) & L(8) & L(6,1,1) & L(4,4)\,. \\
 & & & L(4,3,1) & L(7,1) & \\
 & & & L(8) & &
\end{array}
$$

This is to be read as meaning that quotienting by the bottom i layers reveals the next layer up as a semi-simple submodule, and should be compared with a typical $N = 2$ projective from [32, pp 169–74].

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